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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

STUDYING STRESS STRAIN STATE OF A SMALL THICKNESS INHOMOGENEOUS SPHERE

Specialty: 2002.01 – Mechanics of deformable solid

Field of science: Mechanics

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GENERAL CHARACTERISTICS OF THE WORK

Rationale and development degree of the topic.

Inhomogeneous materials are widely used in various fields of engineering. The assumption on homogeneity of the material ignores its many important mechanical properties. It should be noted that one of the main properties of materials that affects the stress-strain state of elastic bodies is its inhomogeneity. Therefore, last years there are a great number of scientific research works studying various aspects of the analysis of inhomogeneous bodies.

In the second half of 1980 years with purposeful change in macroscopic properties of the material, a new leap was achived by applying new technologies for production of materials. This was implemented by developing a class of composite materials connecting two or multi-phase materials mainly with contradicting properties to FGM conception.

The term FGM was created in the late 1980s. in Japan as description of a class of engineering materials illustrating inhomogenous microstructure and properties. The main goal of implementation of FGM is to improve operational properties of structures subjected to thermal and mechanical effects. Direct choice of such various materials increases the level of residual stress arising because of serious discrepancy of properties in the contact fields between phases and materials and this in its turn, leads to degradation of the material. İn order to minimize the effect of these stresses, the composition of FGM is structured so that all material properties of the assembly changes continuously from one phase component to another one.

One of the main approaches in studying inhomogeneous bodies is the assumption that the material has mechanical properties expressed by elementary functions (hyperbolic, exponential, linear, quadratic, and so on). This approach enables to use classic methods in solving appropriate problems of mechanics, to stimulate and analyze the inhomogeneity. As a result the determined analytic solutions in many cases can be accepted as a reference for solving more complex problems.

Giving the material properties in the form of elementary functions, opens up a wide range of possibilities for exact analysis of inhomogeneous bodies.

Shells and plates are constructions widely used in engineering constructions. The theory of shells in a field that develops analysis methods of thin-walled constructions of modern mechanics. The study of inhomogeneous shells takes a special place in theory of shells. Complex processes occurring during deformation of inhomogeneous shells, leads to emergence of applied theories based on certain assumptions.

The study of application field of the existing applied theories for inhomogeneous shells and creation of a more exact applied theory makes necessary to study the theory of elasticity of inhomogeneous shells on the basis of three-dimensional equations.

Some problems related to the analysis of stress-strain state of inhomogeneous shells can be well defined only on the basis of the equations of elasticity theory. It should be noted that the study of inhomogeneous shells on the basis of the equations of elasticity theory takes their mechanical and geometrical structures into account more adequately. The study of inhomogeneous shells on the basis of the equations of elasticity theory is connected with many mathematical difficulties. But, from physical point of view this creates a new quality and quantity effect.

Object and subject of the study. Application of asymptotic numerical methods to the study of stress-strain state of a small thickness inhomogeneous sphere.

Goals and objectives of the study. To study the elasticity theory problems symmetric with respect to an axis when different boundary conditions are given on a lateral surface of a radial inhomogeneous small thickness sphere whose elasticity modules change with respect to the radius; to determine the stress-strain state of a radial inhomogeneous sphere.

Research methods. The technigue of the research is based on the method of homogeneous solutions, on the method of asymptotic integration of elasticity theory equations, on the method of appropriate solution of the system of partial differential equations.

The main theses to be defended.

To construct homogeneous and inhomogeneous solutions for an elasticity theory problem symmetric with respect to an axis and when various boundary conditions are given on a lateral surface of a small thickness radial inhomogeneous sphere.

To determine the character of stress-strain state of a small thickness radial inhomogeneous sphere corresponding to these solutions.

To obtain asymptotic formulas enabling to calculate the stress-strain state of an inhomogeneous sphere.

Scientific novelty of the study. The main obtained results in the work are the followings:

-An elasticity theory problem with respect to the axis was studied for a small thickness radial inhomogeneous sphere, homogeneous and inhomogeneous solutions were built. It was shown that when the lateral surface of the sphere is free of stress, the homogeneous solution consists of the sum of the extended, simple boundary effect character and boundary layer character solutions. The character of the stress-strain state corresponding to these solutions was determined. Asymptotic formulas for calculating the three-dimensional stress-strain state of the sphere were obtained.

-It was determined that when the lateral surface of the sphere is closed and homogeneous mixed boundary conditions are given on the lateral surface, the homogeneous solutions of the considered problems consist of only boundary layer character solutions.

-A torsion problem for a radial inhomogeneous small thickness sphere was studied. It was shown that when the lateral surface of the sphere is free of tension, the homogeneous solution consists of the sum of the extended and boundary layer character solutions. It was

shown that when the lateral surface of the sphere is closed, the torsion problem has only a baundary layer character solution.
-A problem of torsional vibrations of a radial inhomogeneous small thickness sphere was studied. The roots of the dispersion equation in small values of the parameter characterizing the thickness of the sphere were studied, asymptotic formulas for displacements and tensions corresponding to these roots were obtained.

Theoretical and practical importance of the study. This work is of theoretical character. By means of asymptotic formulas obtained for the components of displacement vector and stress tensor it is possible to calculate the stress-strain state of a radial inhomogeneous sphere. By means of the obtained solutions one can estimate the application area of various applied theories existing for a sphere and contruct more exact applied problems for a radial inhomogeneous sphere.

Approbation and application. The results of the dissertation work were discussed at the International conference “Theoretical and applied problems of mathematics” (Sumgait, 2012), at the I Internatinal science and engineering conference (Baku 2018), at the XXXIX International Scientific Practical conference “Advances in science and technology” (Moscow 2021), at the International conference “Theoretical and applied problems of Mathemtics” (Sumgait 2021), in the seminar of the chair of “Mathematics and statistics” of Azerbaijan State University of Economy.

Author’s personal contribution. Except the statement of some problems the goal of the study, choice of the direction and all the obtained results belong to the author.

Author’s publications. 5 papers in the editions recommended by Higher Attestation Committee at the President of the Republic of Azerbaijan, conference materials – 4.

The name of the organization where the dissertation work was executed. The work was performed at the department of "General technical disciplines and technology" of the Ganja State University.

Structure and volume of the dissertation (in signs indicating the volume of each structural subdivision separately). The dissertation consists of an introduction, two chapters, a conclusion and a list of references, 126 pages. Total volume of the dissertation work 215277 signs (title page 339 signs, contents 2621 signs, introduction 23276 signs, chapter I 104000 signs, chapter II 84000 signs, conclusion 1041 signs). The dissertation work contains 1 figure, 10 graphs and a list of references with 96 names.

THE MAIN CONTENT OF THE WORK

The introduction contains the rationale of the work, review of scientific works related to the topic, the goal of the work and brief content of the work.

Chapter I is called ”Asymptotic analysis of an elasticity theory problem with respect to the axis for a radial inhomogeneous sphere“ and deals with asymptotic theory of a small thickness radial inhomogeneous sphere.

In **1.1** the statement of a boundary value problem is given for a radial inhomogeneous small thickness sphere. An elasticity theory problem symmetric with respect to an axis is considered for a radial inhomogeneous isotropic sphere. In the spherical coordinate system r, θ, φ the sphere has the volume

$$\Gamma = \{r \in [r_1; r_2], \theta \in [\theta_1; \theta_2],\ \varphi \in [0, 2\pi]\}$$

and the poles $0, \pi$ are not contained in the sphere (fig. 1). It is assumed that elasticity modulus change with respect to the radius by the linear law:

$$G(r) = G_* r,\ \lambda(r) = \lambda_* r$$

In the spherical coordinate system, the expression of the displacement vector of the system of balance equations by the components $u_\rho = u_\rho(\rho;\theta)$, $u_\theta = u_\theta(\rho;\theta)$, $u_\varphi = u_\varphi(\rho;\theta)$ is as follows:

$$\begin{cases} (2G_0+\lambda_0)\dfrac{\partial^2 u_\rho}{\partial\rho^2}+2\varepsilon(2G_0+\lambda_0)\dfrac{\partial u_\rho}{\partial\rho}-4\varepsilon^2 G_0 u_\rho - \\ -3\varepsilon^2 G_0\left(\dfrac{\partial u_\theta}{\partial\theta}+u_\theta ctg\,\theta\right)+\varepsilon(G_0+\lambda_0)\left(\dfrac{\partial u_\theta}{\partial\rho}ctg\,\theta+\dfrac{\partial^2 u_\theta}{\partial\theta\partial\rho}\right)+ \\ \varepsilon^2 G_0\left(\dfrac{\partial^2 u_\rho}{\partial\theta^2}+\dfrac{\partial u_\rho}{\partial\theta}ctg\,\theta\right)=0, \end{cases} \quad (1)$$

$$\begin{cases} G_0\dfrac{\partial^2 u_\theta}{\partial\rho^2}+2\varepsilon G_0\dfrac{\partial u_\theta}{\partial\rho}+(5G_0+2\lambda_0)\varepsilon^2\dfrac{\partial u_\rho}{\partial\theta}+ \\ +\varepsilon(G_0+\lambda_0)\dfrac{\partial^2 u_\rho}{\partial\rho\partial\theta}+\varepsilon^2(2G_0+\lambda_0)\times \\ \times\left(\dfrac{\partial u_\theta}{\partial\theta}ctg\,\theta+\dfrac{\partial^2 u_\theta}{\partial\theta^2}-u_\theta ctg^2\theta\right)-\varepsilon^2(\lambda_0+3G_0)u_\theta=0. \end{cases} \quad (2)$$

$$\frac{\partial^2 u_\varphi}{\partial\rho^2}+2\varepsilon\frac{\partial u_\varphi}{\partial\rho}+$$

$$+\varepsilon^2\left(\frac{\partial^2 u_\varphi}{\partial\theta^2}+\frac{\partial u_\varphi}{\partial\theta}ctg\,\theta-\frac{\cos 2\theta}{\sin^2\theta}-u_\varphi-3u_\varphi\right)=0, \quad (3)$$

Here $\rho=\dfrac{1}{\varepsilon}\ln\left(\dfrac{r}{r_0}\right)$ is a new pure variable; $r_0=\sqrt{r_1 r_2}$; $\varepsilon=\dfrac{1}{2}\ln\left(\dfrac{r_2}{r_1}\right)$; $\rho\in[-1,1]$. Note that ε is a small parameter characterizing the thickness of the sphere.

Assume that the lateral surface of the sphere is free from load:

$$\begin{cases} \left[(2G_0+\lambda_0)\dfrac{\partial u_\rho}{\partial \rho} + \varepsilon\lambda_0 \left(u_\theta ctg\,\theta + 2u_\rho + \dfrac{\partial u_\theta}{\partial \theta} \right) \right] \Bigg|_{\rho=\pm 1} = 0, & (4) \\ G_0 \left[\dfrac{\partial u_\theta}{\partial \rho} + \varepsilon \left(\dfrac{\partial u_\rho}{\partial \theta} - u_\theta \right) \right] \Bigg|_{\rho-\pm 1} = 0. & (5) \end{cases}$$

$$G_0 \left(\frac{\partial u_\varphi}{\partial \rho} - \varepsilon u_\varphi \right) \Bigg|_{\rho=\pm 1} = 0. \qquad (6)$$

Here $G_0 = \dfrac{G_* r_0}{t}$, $\lambda_0 = \dfrac{\lambda_* r_0}{t}$ are pure variables; t is a characteristical quantity with elasticity modulus dimension.

Assume that we are given the boundary conditions

$$\sigma_{\theta\theta}\Big|_{\theta=\theta_n} = f_{1n}(\rho), \ \ \sigma_{\rho\theta}\Big|_{\theta=\theta_n} = f_{2n}(\rho) \qquad (7)$$

$$\sigma_{\varphi\theta}\Big|_{\theta=\theta_s} = f_s(\rho)$$

balancing the sphere in the base of the sphere.

The problem (3), (6),(8) characterizes the torsion of the sphere.

In **1.2** the problem (1), (2), (4),(5) is studied.

The solution of the boundary value problem (1), (2), (4), (5) is sought in the form

$$u_\rho(\rho;\theta) = a(\rho)m(\theta); \ \ u_\theta(\rho;\theta) = d(\rho)m'(\theta) \qquad (9)$$

Here the function $m(\theta)$ is the solution of the Legendre equation

$$m''(\theta) + ctg\,\theta m'(\theta) + \left(z^2 - \frac{1}{4} \right) m(\theta) = 0 \qquad (10)$$

Having substituted (9) in (1),(2),(4),(5) according to (10) we obtain:

$$
\begin{cases}
(2G_0+\lambda_0)(a''(\rho)+2\varepsilon a'(\rho))-\varepsilon^2 G_0\left(z^2+\dfrac{15}{4}\right)a(\rho)- \\
-\varepsilon(G_0+\lambda_0)\left(z^2-\dfrac{1}{4}\right)d'(\rho)+3\varepsilon^2 G_0\left(z^2-\dfrac{1}{4}\right)d(\rho)=0, & (11) \\
G_0(d''(\rho)+2\varepsilon d'(\rho))-\varepsilon^2\left(G_0+(2G_0+\lambda_0)\left(z^2-\dfrac{1}{4}\right)\right)d(\rho)+ \\
+\varepsilon^2(5G_0+2\lambda_0)a(\rho)+\varepsilon(G_0+\lambda_0)a'(\rho)=0, & (12)
\end{cases}
$$

$$
\begin{cases}
\left.\left[(2G_0+\lambda_0)a'(\rho)+\varepsilon\lambda_0\left(2a(\rho)-\left(z^2-\dfrac{1}{4}\right)d(\rho)\right)\right]\right|_{\rho=\pm 1}=0, & (13) \\
G_0\left.\left[d'(\rho)+\varepsilon(a(\rho)-d(\rho))\right]\right|_{\rho=\pm 1}=0. & (14)
\end{cases}
$$

The solution of the system (11), (12) is as follws:

$$a(\rho)=e^{-\varepsilon\rho}[p_1 e^{\varepsilon s_1\rho}A_1+p_1 e^{-\varepsilon s_1\rho}A_2+p_2 e^{\varepsilon s_2\rho}A_3+p_2 e^{-\varepsilon s_2\rho}A_4], \qquad (15)$$

$$d(\rho)=e^{-\varepsilon\rho}[t_1 e^{\varepsilon s_1\rho}A_1+q_1 e^{-\varepsilon s_1\rho}A_2+t_2 e^{\varepsilon s_2\rho}A_3+q_2 e^{-\varepsilon s_2\rho}A_4], \qquad (16)$$

Here the s_k are the roots of the equation

$$(2G_0+\lambda_0)G_0 s^4-\left[\left(z^2-\frac{1}{4}\right)2G_0(2G_0+\lambda_0)+G_0(10G_0+3\lambda_0)\right]s^2+$$

$$+\left(z^2-\frac{1}{4}\right)^2 G_0(2G_0+\lambda_0)-2\left(z^2-\frac{1}{4}\right)G_0^2+2G_0(6G_0+\lambda_0)=0$$

$$p_k=G_0 s_k^2-\left(z^2-\frac{1}{4}\right)(2G_0+\lambda_0)-2G_0;$$

$$t_k=-(G_0+\lambda_0)s_k-(4G_0+\lambda_0);\; q_k=(G_0+\lambda_0)s_k-(4G_0+\lambda_0);$$

Having substituted (15),(16) in the homogeneous boundary conditions (13), (14) from the existence of non-trivial solution of the obtained system of homogeneous linear algebraic equations we determine the characteristic equation

$$\Delta_1(z,\varepsilon)=(Q_{21}D_{22}-Q_{22}D_{21})\cdot(Q_{12}D_{11}-$$
$$-Q_{11}D_{12})sh^2(\varepsilon(s_1+s_2))+(Q_{21}D_{12}-Q_{12}D_{21})\cdot$$
$$\cdot(Q_{11}D_{22}-Q_{22}D_{11})sh^2\cdot(\varepsilon(s_1-s_2))=0, \qquad (17)$$

Here

$$D_{1k}=-\lambda_0 s_k^2+(\lambda_0-2G_0)s_k-\left(z^2-\frac{1}{4}\right)(2G_0+\lambda_0)+6G_0+2\lambda_0;$$

$$D_{2k}=-\lambda_0 s_k^2+(2G_0-\lambda_0)s_k-\left(z^2-\frac{1}{4}\right)(2G_0+\lambda_0)+6G_0+2\lambda_0;$$

$$Q_{1k}=(2G_0+\lambda_0)G_0 s_k^3+G_0(\lambda_0-2G_0)s_k^2-\left[2G_0(2G_0+\lambda_0)+\right.$$
$$\left.+\left(z^2-\frac{1}{4}\right)G_0(4G_0+3\lambda_0)\right]s_k+\left(z^2-\frac{1}{4}\right)4G_0(G_0+\lambda_0)+2G_0(2G_0-\lambda_0);$$

$$Q_{2k}=-(2G_0+\lambda_0)G_0 s_k^3+G_0(\lambda_0-2G_0)s_k^2+\left[2G_0(2G_0+\lambda_0)+\right.$$
$$\left.+\left(z^2-\frac{1}{4}\right)G_0(4G_0+3\lambda_0)\right]s_k+4G_0(G_0+\lambda_0)\left(z^2-\frac{1}{4}\right)+2G_0(2G_0-\lambda_0);(k=1;2).$$

In **1.3** as $\varepsilon\to 0$ the asymptotic analysis of the roots of the characteristic equation (17) is conducted.

Theorem 1. As $\varepsilon\to 0$ the set of roots $\Lambda(z,\varepsilon)$ of equation (17) is a denumerable set and the equality

$$\Lambda(z,\varepsilon)=\Lambda_1(z)\bigcup\Lambda_2(z,\varepsilon)\bigcup\Lambda_3(z,\varepsilon)$$

is valid:

1. The set $\Lambda_1(z)$ consists of the roots $z=\pm\frac{3}{2}$.

2. The set $\Lambda_2(z;\varepsilon)$ consists of the four roots

$$z_k=\varepsilon^{-\frac{1}{2}}\left(\beta_{0k}+\varepsilon\beta_{1k}+...\right) \qquad (18)$$

of order $O\left(\varepsilon^{-\frac{1}{2}}\right)$.

Here

$$\beta_{0k} = \chi_{0k}\,, \quad \beta_{1k} = \frac{1}{4h_1\beta_{0k}}\left(h_0 - \frac{h_1(5h_2h_3 - 2h_1h_6)}{5h_2^2}\right), \ldots (k = \overline{1;4})$$

$$\chi_{0k}^4 + 3\frac{h_1}{h_2} = 0\,.$$

3. The set $\Lambda_3(z;\varepsilon)$ consists of denumerable numbers roots

$$z_k = \frac{\gamma_{0k}}{\varepsilon} + O(\varepsilon) \qquad (19)$$

of order $O\left(\varepsilon^{-1}\right)$.

In (19), γ_{0k} are the solutions of the equation $sh^2(2\gamma_{0k}) - 4\gamma_{0k}^2 = 0$.

Asymptotic solutions corresponding to the roots of the characteristic equation (17) are determined:

1)

a) $z = -\dfrac{3}{2}$

$$u_\rho^{(1)} = B\left(\cos\theta \ln\left(ctg\frac{\theta}{2}\right) - 1\right), \qquad (21)$$

$$u_\theta^{(1)} = -B\left(\sin\theta \ln\left(ctg\frac{\theta}{2}\right) + ctg\,\theta\right), \qquad (22)$$

b) $z = \dfrac{3}{2}$

$$u_\rho^{(1)} = M\cos\theta; \quad u_\theta = -M\sin\theta, \qquad (23)$$

$$\sigma_{\rho\rho}^{(1)} = \sigma_{\theta\theta}^{(1)} = \sigma_{\varphi\varphi}^{(1)} = \sigma_{\rho\theta}^{(1)} = 0.$$

The solution (23) characterizes the motion of the sphere as an absolute solid.

2) $$z = \varepsilon^{-\frac{1}{2}}\left(\beta_{0j} + \varepsilon\beta_{1j} + \ldots\right). \qquad (24)$$

$$u_{\rho}^{(2)}(\rho,\theta)=\sum_{j=1}^{4}E_j u_{\rho j}^{(2)}(\rho,\theta), \qquad (25)$$

$$u_{\theta}^{(2)}(\rho,\theta)=\sum_{j=1}^{4}E_j u_{\theta j}^{(2)}(\rho,\theta), \qquad (26)$$

here

$$u_{\rho j}^{(2)}(\rho,\theta)=\left[1+\varepsilon\left(-\frac{\lambda_0}{2(G_0+\lambda_0)}\rho-\frac{\lambda_0}{2(2G_0+\lambda_0)}\beta_{0j}^2\rho^2\right)+O\left(\varepsilon^2\right)\right]m_j(\theta),$$

$$u_{\theta j}^{(2)}(\rho,\theta)=\varepsilon\left[-\rho+\frac{2G_0+3\lambda_0}{2(G_0+\lambda_0)}\frac{1}{\beta_{0j}^2}+O(\varepsilon)\right]m_j'(\theta),$$

3)

a) $$z_k=\varepsilon^{-1}(\gamma_{0k}+\varepsilon\gamma_{1k}+\ldots). \qquad (27)$$

$$u_{\rho}^{(3,1)}(\rho,\theta)=\sum_{k=1}^{\infty}D_k u_{\rho k}^{(3,1)}(\rho,\theta), \qquad (28)$$

$$u_{\theta}^{(3,1)}(\rho,\theta)=\sum_{k=1}^{\infty}D_k u_{\theta k}^{(3,1)}(\rho,\theta), \qquad (29)$$

here

$$u_{\rho k}^{(3,1)}(\rho,\theta)=\gamma_{0k}\left[\left(\left(G_0+\lambda_0\right)\gamma_{0k}ch\gamma_{0k}+\left(2G_0+\lambda_0\right)sh\gamma_{0k}\right)sh(\gamma_{0k}\rho)-\right.$$
$$\left.-\left(G_0+\lambda_0\right)\gamma_{0k}\rho sh\gamma_{0k}ch(\gamma_{0k}\rho)+O(\varepsilon)\right]m_k(\theta),$$

$$u_{\theta k}^{(3,1)}(\rho,\theta)=\varepsilon\left[\left(\left(G_0+\lambda_0\right)\gamma_{0k}-G_0 sh\gamma_{0k}\right)ch(\gamma_{0k}\rho)-\right.$$
$$\left.-\left(G_0+\lambda_0\right)\rho\gamma_{0k}sh\gamma_{0k}sh(\gamma_{0k}\rho)+O(\varepsilon)\right]m_k'(\theta),$$

γ_{0k} are the roots of the equation

$$sh(2\gamma_{0k})+2\gamma_{0k}=0$$

b) $$z_i=\varepsilon^{-1}(\gamma_{0i}+\varepsilon\gamma_{1i}+\ldots), \qquad (30)$$

$$u_{\rho}^{(3,2)}(\rho,\theta)=\sum_{i=1}^{\infty}F_i u_{\rho i}^{(3,2)}(\rho,\theta), \qquad (31)$$

$$u_{\theta}^{(3,2)}(\rho,\theta)=\sum_{i=1}^{\infty}F_i u_{\theta i}^{(3,2)}(\rho,\theta), \tag{32}$$

here

$$u_{\rho i}^{(3,2)}(\rho,\theta)=\gamma_{0i}\left[\left((G_0+\lambda_0)\gamma_{0i}sh\gamma_{0i}+(2G_0+\lambda_0)ch\gamma_{0i}\right)ch(\gamma_{0i}\rho)-\right.$$
$$\left.-(G_0+\lambda_0)\gamma_{0i}\rho ch\gamma_{0i}sh(\gamma_{0i}\rho)+O(\varepsilon)\right]m_i(\theta),$$
$$u_{\theta i}^{(3,2)}(\rho,\theta)=\varepsilon\left[\left((G_0+\lambda_0)\gamma_{0i}sh\gamma_{0i}-G_0ch\gamma_{0i}\right)sh(\gamma_{0i}\rho)-\right.$$
$$\left.-(G_0+\lambda_0)\rho\gamma_{0i}ch\gamma_{0i}ch(\gamma_{0i}\rho)+O(\varepsilon)\right]m_i'(\theta),$$

γ_{0i} are the roots of the equation

$$sh(2\gamma_{0i})-2\gamma_{0i}=0.$$

For the general solution of the problem (1), (2), (4), (5) we get:

$$u_{\rho}(\rho;\theta)=u_{\rho}^{(1)}+$$
$$+\sum_{j=1}^{4}E_j u_{\rho j}^{(2)}(\rho;\theta)+\sum_{k=1}^{\infty}D_k u_{\rho k}^{(3,1)}(\rho;\theta)+\sum_{i=1}^{\infty}F_i u_{\rho i}^{(3,2)}(\rho;\theta), \tag{33}$$

$$u_{\theta}(\rho;\theta)=u_{\theta}^{(1)}+$$
$$+\sum_{j=1}^{4}E_j u_{\theta j}^{(2)}(\rho;\theta)+\sum_{k=1}^{\infty}D_k u_{\theta k}^{(3,1)}(\rho;\theta)+\sum_{i=1}^{\infty}F_i u_{\theta i}^{(3,2)}(\rho;\theta), \tag{34}$$

In **1.4** the stress-strain state corresponding to the homogeneous solutions is analyzed, the character of the solutions is determined. It is shown that the solution (21), (22) is equivalent to the principle vector P of the stresses acting on the section $\theta=const$:

$$P=-4\pi G_0 sh(2\varepsilon)B \tag{35}$$

The stress state determined by the solutions (25), (26), (28), (29), (31), (32) is self balanced in the section $\theta=const$.

(21), (22) is an extended solution. The stress state corresponding to the solution (21), (22) is equivalent to the principle vector of the forces applied to the arbitrary section $\theta=const$ of the sphere. The stress state corresponding to the solution (25),(26) determines the boundary effect in applied theory of shells. The stress state determined by the solutions (25), (26) is equivalent to the bending moment $M_{əy.}$ and to the cutting force $Q_{kəs.}$. The solutions (25), (26) determine the principal parts of the bending moment and cutting forces. The solutions (21), (22), (25), (26) determine the internal stress strain state of the sphere. The solutions (28), (29), (31), (32) are of boundary layer character. The first term of these solutions is equivalent to the Saint-Venant effect in theory of inhomogeneous plates. The boundary layer character solutions have not been determined in the classic Kirchhoff-Liav theory of shells. When moving away from the conic sections $\theta = \theta_j \ (j = 1;2)$ the solutions (25),(26),(28), (29), (31),(32) damp by the exponential law.

The stress-strain state of an inhomogeneous sphere consists of the sum of extended simple boundary effect character and boundary layer chacarter solutions.

In **1.5** the problem of satisfaction of boundary conditions on the base of the sphere is considered.

The expression of the constant B by means of the principal vector P is determined by means of the formula

$$B = -\frac{P}{4\pi G_0 sh(2\varepsilon)}$$

The constants E_j, D_k, F_i are determined from the boundary conditions (7) given in the base of the sphere. Since the homogeneous solutions satisfy the homogeneous conditions and the balance equations given in the lateral surface according to the Lagrange variation principle we can write:

$$\sum_{j=1}^{2}\int_{-1}^{1}\left[\left(\sigma_{\theta\theta} - f_{1j}(\rho)\right)\delta u_\theta + \left(\sigma_{\rho\theta} - f_{2j}(\rho)\right)\delta u_\rho\right]\Big|_{\theta=\theta_j} e^{2\varepsilon\rho}\, d\rho = 0. \qquad (36)$$

Considering $\delta E_j, \delta D_k, \delta F_i$ as independent variations, from (36) we obtain the following system of linear algebraic equations:

$$\sum_{k=1}^{4} l_{jk} E_{k0} = \tau_{j0}; \quad (j = \overline{1,4}), \qquad (37)$$

$$l_{jk}=\left(\frac{1}{\sin\theta_2}-\frac{1}{\sin\theta_1}\right)\frac{8(G_0+\lambda_0)}{3(2G_0+\lambda_0)}\frac{\beta_{0k}^2}{\sqrt[4]{\beta_{0j}^2\beta_{0k}^2}}\left(\sqrt{-\beta_{0k}^2}-\sqrt{-\beta_{0j}^2}G_0\right);$$

$$\tau_{j0}=\frac{1}{\sqrt[4]{-\beta_{0j}^2}}\sum_{n=1}^{2}\frac{1}{\sqrt{\sin\theta_n}}\times$$

$$\times\left[f_{2n}^*+(-1)^n\sqrt{-\beta_{0j}^2}\int_{-1}^{1}f_{1n}(\rho)\left(\frac{2G_0+3\lambda_0}{2(G_0+\lambda_0)}\frac{1}{\beta_{0j}^2}-\rho\right)d\rho\right].$$

$$f_{2n}^*=\lim_{\varepsilon\to 0}\frac{f_{2n}^{(1)}}{\sqrt{\varepsilon}}.$$

$$E_j=E_{j0}+\varepsilon E_{j1}+\ldots \tag{38}$$

$$\sum_{k=1}^{\infty}M_{kj}^{(1)}D_{k0}=\tau_{j1};\quad (j=1,2,\ldots) \tag{39}$$

$$M_{kj}^{(1)}=2G_0(G_0+\lambda_0)\left(\frac{1}{\sin\theta_2}-\frac{1}{\sin\theta_1}\right)\frac{\gamma_{0k}}{\sqrt[4]{\gamma_{0k}^2\gamma_{0j}^2}}\times$$

$$\times\left(\gamma_{0j}\sqrt{-\gamma_{0k}^2}\int_{-1}^{1}\left[\gamma_{0k}ch\gamma_{0k}sh(\gamma_{0k}\rho)-\gamma_{0k}\rho sh\gamma_{0k}ch(\gamma_{0k}\rho)\right]\times\right.$$

$$\times\left\{\left[(G_0+\lambda_0)\gamma_{0j}ch\gamma_{0j}+(2G_0+\lambda_0)sh\gamma_{0j}\right]sh(\gamma_{0j}\rho)-\right.$$

$$\left.-(G_0+\lambda_0)\gamma_{0j}\rho sh\gamma_{0j}ch(\gamma_{0j}\rho)\right\}d\rho-\gamma_{0k}\sqrt{-\gamma_{0j}^2}\times$$

$$\times\int_{-1}^{1}\left[(\gamma_{0k}ch\gamma_{0k}-sh\gamma_{0k})ch(\gamma_{0k}\rho)-\gamma_{0k}\rho sh(\gamma_{0k}\rho)sh\gamma_{0k}\right]\times$$

$$\times\left\{\left[(G_0+\lambda_0)\gamma_{0j}ch\gamma_{0j}-G_0sh\gamma_{0j}\right]ch(\gamma_{0j}\rho)-\right.$$

$$\left.\left.-(G_0+\lambda_0)\gamma_{0j}\rho sh\gamma_{0j}sh(\gamma_{0j}\rho)\right\}d\rho\right)$$

$$\tau_{j1} = \sum_{n=1}^{2} \frac{1}{\sqrt{\sin\theta_n}} \left\langle (-1)^n \frac{\sqrt{-\gamma_{0j}^2}}{\sqrt[4]{-\gamma_{0j}^2}} \times \right.$$

$$\times \int_{-1}^{1} f_{1n}^*(\rho) \left\{ \left[(G_0+\lambda_0)\gamma_{0j} ch\gamma_{0j} - G_0 sh\gamma_{0j} \right] ch(\gamma_{0j}\rho) - \right.$$

$$\left. -(G_0+\lambda_0)\gamma_{0j}\rho sh(\gamma_{0j}\rho) sh\gamma_{0j} \right\} d\rho +$$

$$+ \frac{\gamma_{0j}}{\sqrt[4]{-\gamma_{0j}^2}} \int_{-1}^{1} f_{2n}(\rho) \left\{ \left[(G_0+\lambda_0)\gamma_{0j} ch\gamma_{0j} + \right. \right.$$

$$\left. \left. + (2G_0+\lambda_0) sh\gamma_{0j} \right] sh(\gamma_{0j}\rho) - (G_0+\lambda_0)\gamma_{0j}\rho sh\gamma_{0j} ch(\gamma_{0j}\rho) \right\} d\rho \Big\rangle,$$

$$\sum_{i=1}^{\infty} N_{ij}^{(1)} F_{i0} = \tau_{j2}; \ (j=1,2,...) \tag{40}$$

$$N_{ij}^{(1)} = 2G_0(G_0+\lambda_0)\left(\frac{1}{\sin\theta_2} - \frac{1}{\sin\theta_1} \right) \frac{\gamma_{0i}}{\sqrt[4]{\gamma_{0i}^2 \gamma_{0j}^2}} \times$$

$$\Bigg(\gamma_{0j}\sqrt{-\gamma_{0i}^2} \int_{-1}^{1} \left[\gamma_{0i} sh\gamma_{0i} ch(\gamma_{0i}\rho) - \gamma_{0i}\rho ch\gamma_{0i} sh(\gamma_{0i}\rho) \right] \times$$

$$\times \left\{ \left[(G_0+\lambda_0)\gamma_{0j} sh\gamma_{0j} + (2G_0+\lambda_0) ch\gamma_{0j} \right] \times \right.$$

$$\left. \times ch(\gamma_{0j}\rho) - (G_0+\lambda_0)\gamma_{0j}\rho ch\gamma_{0j} sh(\gamma_{0j}\rho) \right\} d\rho$$

$$- \gamma_{0i}\sqrt{-\gamma_{0j}^2} \int_{-1}^{1} \left[(\gamma_{0i} sh\gamma_{0i} - ch\gamma_{0i}) sh(\gamma_{0i}\rho) - \gamma_{0i}\rho ch(\gamma_{0i}\rho) ch\gamma_{0i} \right] \times$$

$$\times \left\{ \left[(G_0+\lambda_0)\gamma_{0j} sh\gamma_{0j} - G_0 ch\gamma_{0j} \right] sh(\gamma_{0j}\rho) - \right.$$

$$\left. -(G_0+\lambda_0)\gamma_{0j}\rho ch\gamma_{0j} ch(\gamma_{0j}\rho) \right\} d\rho \Bigg),$$

$$\tau_{j2} = \sum_{n=1}^{2} \frac{1}{\sqrt{\sin\theta_n}} \left\langle (-1)^n \frac{\sqrt{-\gamma_{0j}^2}}{\sqrt[4]{-\gamma_{0j}^2}} \times \right.$$

$$\times\int_{-1}^{1} f_{1n}^{*}(\rho)\Big\{\big[(G_0+\lambda_0)\gamma_{0j}sh\gamma_{0j}-G_0ch\gamma_{0j}\big]sh(\gamma_{0j}\rho)-$$

$$-(G_0+\lambda_0)\gamma_{0j}\rho sh(\gamma_{0j}\rho)sh\gamma_{0j}\Big\}d\rho+$$

$$+\frac{\gamma_{0j}}{\sqrt[4]{-\gamma_{0j}^2}}\int_{-1}^{1} f_{2n}(\rho)\Big\{\big[(G_0+\lambda_0)\gamma_{0j}sh\gamma_{0j}+(2G_0+\lambda_0)ch\gamma_{0j}\big]ch(\gamma_{0j}\rho)$$

$$-(G_0+\lambda_0)\gamma_{0j}\rho ch\gamma_{0j}sh(\gamma_{0j}\rho)\Big\}d\rho\Big\rangle,$$

$$D_k=\varepsilon(D_{k0}+\varepsilon D_{k1}+...), \tag{41}$$

$$F_i=\varepsilon(F_{i0}+\varepsilon F_{i1}+...)\quad, \tag{42}$$

$$f_{1n}^{*}(\rho)=f_{1n}(\rho)+\frac{P}{2\pi\sin^2\theta\, sh2\varepsilon}.$$

$$f_{2n}^{(1)}=\int_{-1}^{1} f_{2n}(\rho)d\rho,\quad f_{2n}^{(2)}=f_{2n}(\rho)-f_{2n}^{(1)};\ (n=1;2).$$

The matrices of the system of linear algebraic equations obtained for determining the constants $E_{jn}, D_{kn}, F_{in}\,(n=1,2,....)$ contained in (38), (41), (42) are the same with the matrices of the systems (37), (39), (40).

In **1.6** special solution satisfying inhomogeneous boundary conditions of the balance equations given on the lateral surface is built according to the first iteration process of the asymptotic integration method.

In **1.7** an elasticity theory problem symmetric with respect to the axis is studied for a an inhomogeneous sphere withsecond kind boundary conditions given on the lateral surface. It is assumed that the lateral surface of the sphere is closed

$$\begin{cases} u_\rho\big|_{\rho=\pm1}=0, & (43)\\ u_\theta\big|_{\rho=\pm1}=0, & (44)\end{cases}$$

and the boundary conditions balancing it are given on the base of the sphere (on conic sections).

Having written (9), (15), (16) in the boundary conditions (43), (44) from the existence of non-trivial solution of the obtained system of homogeneous linear algebraic equations we obtain the characteristic equation

$$\begin{aligned}\Delta_2(z;\varepsilon) &= (p_1q_2 - p_2q_1)(t_1p_2 - p_1t_2)sh^2(\varepsilon(s_1+s_2))+\\ &+(p_1t_2 - p_2q_1)(p_1q_2 - p_2t_1)sh^2(\varepsilon(s_1 - s_2)) = 0,\end{aligned} \tag{45}$$

Theorem 2. As $\varepsilon \to 0$ the set of roots of the equation (45) consists of denumerable number roots $z_k = \dfrac{\delta_{0k}}{\varepsilon} + O(\varepsilon)$ and the δ_{0k} are the solutions of the equation

$$sh^2(2\delta_{0k}) - \frac{4(G_0+\lambda_0)^2}{(3G_0+\lambda_0)^2}\delta_{0k}^2 = 0$$

The following asymptotic solutions correspond to the indicated roots of the characteristic equation (45):

a)
$$z_k = \varepsilon^{-1}(\delta_{0k} + \varepsilon\delta_{1k} + ...), \tag{46}$$

$$u_\rho^{(3,1)}(\rho;\theta) = \sum_{k=1}^{\infty} B_{1k} u_{\rho k}^{(3,1)}(\rho;\theta), \tag{47}$$

$$u_\theta^{(3,1)}(\rho;\theta) = \sum_{k=1}^{\infty} B_{1k} u_{\theta k}^{(3,1)}(\rho;\theta), \tag{48}$$

here

$$\begin{aligned}u_{\rho k}^{(3,1)}(\rho;\theta) &= \delta_{0k}[(G_0+\lambda_0)\delta_{0k} sh\delta_{0k} sh(\delta_{0k}\rho) -\\ &- ch\delta_{0k}((G_0+\lambda_0)\delta_{0k}\rho ch(\delta_{0k}\rho) - (3G_0+\lambda_0)sh(\delta_{0k}\rho)) + O(\varepsilon)]m_k(\theta),\end{aligned}$$

$$\begin{aligned}u_{\theta k}^{(3,1)}(\rho;\theta) &= \varepsilon(G_0+\lambda_0)\delta_{0k}[sh\delta_{0k} ch(\delta_{0k}\rho) -\\ &- \rho ch\delta_{0k} sh(\delta_{0k}\rho) + O(\varepsilon)]m_k^{'}(\theta),\end{aligned}$$

the δ_{0k} are the roots of the equation

$$sh(2\delta_{0k}) - \frac{2(G_0 + \lambda_0)}{3G_0 + \lambda_0}\delta_{0k} = 0,$$

b) $$z_k = \varepsilon^{-1}(\delta_{0i} + \varepsilon\delta_{1i} + ...), \qquad (49)$$

$$u_{\rho}^{(3,2)}(\rho;\theta) = \sum_{i=1}^{\infty} B_{2i} u_{\rho i}^{(3,2)}(\rho;\theta), \qquad (50)$$

$$u_{\theta}^{(3,2)}(\rho;\theta) = \sum_{i=1}^{\infty} B_{2i} u_{\theta i}^{(3,2)}(\rho;\theta), \qquad (51)$$

here

$$u_{\rho i}^{(3,2)}(\rho;\theta) = \delta_{0i}[(G_0 + \lambda_0)\delta_{0i} ch\delta_{0i} ch(\delta_{0i}\rho) -$$
$$- sh\delta_{0i}((G_0 + \lambda_0)\delta_{0i}\rho sh(\delta_{0i}\rho) - (3G_0 + \lambda_0) ch(\delta_{0i}\rho)) + O(\varepsilon)] m_i(\theta),$$
$$u_{\theta i}^{(3,2)}(\rho;\theta) = \varepsilon(G_0 + \lambda_0)\delta_{0i}[ch\delta_{0i} sh(\delta_{0i}\rho) -$$
$$- \rho sh\delta_{0i} ch(\delta_{0i}\rho) + O(\varepsilon)] m_i^{'}(\theta),$$

the δ_{0i} are the roots of the equation

$$sh(2\delta_{0i}) + \frac{2(G_0 + \lambda_0)}{3G_0 + \lambda_0}\delta_{0i} = 0.$$

Is moving away from the conical sections $\theta = \theta_j$ $(j = 1;2)$ to the inside of the sphere, the solutions (47), (48), (50), (51) decay exponentially. The noted solutions are of boundary layer character and were not determined in the classic Kirchhof-Liav theory. The first terms of the solutions (47), (48), (50), (51) are equivalent to the Saint-Venant boundary effect in theory of inhomogeneous plates.

As a result of satisfaction of boundary conditions given on the base of the sphere, we obtain an infinite system of linear algebraic equations to determine the constants contained in (47), (48), (50), (51.

In **1.8** we study an elasticity theory problem symmetric with respect to the axis and with mixed boundary conditions on the lateral surface.

It is assumed that the mixed boundary conditions

$$\begin{cases} u_\rho\big|_{\rho=\pm 1} = 0, & (52) \\ \sigma_{\rho\theta}\big|_{\rho=\pm 1} = 0, & (53) \end{cases}$$

are given on the lateral surface of the sphere, the boundary conditions balancing the sphere are given on the base of the sphere. From the existence of nontrivial solution of the system of homogeneous linear algebraic equations obtained when writting (9), (15), (16) in boundary conditions (52), (53) we determine the characteristic equation

$$\Delta_3(z;\varepsilon) = \left(p_1 D_{22} - p_2 D_{21}\right)\left(p_2 D_{11} - p_1 D_{12}\right) sh^2\left(\varepsilon\left(s_1 + s_2\right)\right) + \\ + \left(p_1 D_{12} - p_2 D_{21}\right)\left(p_1 D_{22} - p_2 D_{11}\right) sh^2\left(\varepsilon\left(s_2 - s_1\right)\right) = 0, \quad (54)$$

Theorem 3. As $\varepsilon \to 0$ the equation (54) has a denumerable number of roots $z_k = \dfrac{\alpha_{0k}}{\varepsilon} + O(\varepsilon)$ and the α_{0k} are the solutions of the equation $sh^2 2\alpha_{0k} = 0$.

The following asymptotic solutions correspond to the roots of the characteristic equation (54) :

a) $$z_k = \varepsilon^{-1}(\alpha_{0k} + \varepsilon\alpha_{1k} + \ldots), \quad (55)$$

$$u_\rho^{(3,1)}(\rho;\theta) = \sum_{k=1}^{\infty} B_{3k} u_{\rho k}^{(3,1)}(\rho;\theta), \quad (56)$$

$$u_\theta^{(3,1)}(\rho;\theta) = \sum_{k=1}^{\infty} B_{3k} u_{\theta k}^{(3,1)}(\rho;\theta), \quad (57)$$

$$u_{\rho k}^{(3,1)}(\rho;\theta) = \alpha_{0k}^3[(G_0 + \lambda_0) ch\alpha_{0k} sh(\alpha_{0k}\rho) + O(\varepsilon)] m_k(\theta),$$

$$u_{\theta k}^{(3,1)}(\rho;\theta) = \varepsilon\alpha_{0k}^2[(G_0 + \lambda_0) ch\alpha_{0k} ch(\alpha_{0k}\rho) + O(\varepsilon)] m_k'(\theta),$$

the α_{0k} are the roots of the equation

$$sh^2\alpha_{0k} = 0.$$

b) $$z_i = \varepsilon^{-1}(\alpha_{0i} + \varepsilon\alpha_{1i} + \ldots), \quad (58)$$

$$u_{\rho}^{(3,2)}(\rho;\theta)=\sum_{i=1}^{\infty}B_{4i}u_{\rho i}^{(3,2)}(\rho;\theta), \tag{59}$$

$$u_{\theta}^{(3,2)}(\rho;\theta)=\sum_{i=1}^{\infty}B_{4i}u_{\theta i}^{(3,2)}(\rho;\theta), \tag{60}$$

here

$$u_{\rho i}^{(3,2)}(\rho;\theta)=\alpha_{0i}^{3}[(G_0+\lambda_0)sh\alpha_{0i}ch(\alpha_{0i}\rho)+O(\varepsilon)]m_i(\theta),$$

$$u_{\theta i}^{(3,2)}(\rho;\theta)=\varepsilon\alpha_{0i}^{2}[(G_0+\lambda_0)sh\alpha_{0i}sh(\alpha_{0i}\rho)+O(\varepsilon)]m_i^{'}(\theta),$$

the α_{0i} are the roots of the equation

$$ch^2\alpha_{0i}=0\,.$$

The solutions (56), (57), (59), (60) are of boundary layer character and the first terms of their expansions with respect to the parameter ε are equivalent to the Saint-Venant boundary effect in theory of inhomogeneous plates.

In **1.9** we consider an elasticity theory problem for a sphere with lateral surface free of stresses and with mixed boundary conditions given on the base. The generalized orthogonality condition is determine according to the Betty theorem and the boundary conditions given on the bases of the sphere are satisfied exactly.

In **1.10** for estimating the influence of materials inhomogeneity on the stress-strain state, the problem for a small thickness radial inhomogeneous and homogeneous isotropic sphere is solved numerically. The results obtained for radially inhomogeneous isotropic spheres are compared. The influence of inhomogeneity on the stess-strain state is estimated

Chapter II is called ”A torsional problem for a small thickness radial inhomogeneous sphere“. In this chapter a torsional problem is studied for a radial inhomogeneous small thickness sphere.

In **2.1** the torsional problen is studied for a radial inhomogeneous small thickness sphere whoses pateral surface is free from load, and whose boundary conditions balancing it are given in its bases.

The solution of the boundary value problem (3), (6) is sought in the form

$$u_\varphi(\rho;\theta) = c(\rho)m'(\theta), \tag{61}$$

Allowing for (10) having written (61) in (3), (6), we get the following result:

$$\begin{cases} c''(\rho) + 2\varepsilon c'(\rho) - \varepsilon^2\left(z^2 + \dfrac{3}{4}\right)c(\rho) = 0, & (62) \\ \left[c'(\rho) - \varepsilon^2 c(\rho)\right]\Big|_{\rho=\pm1} = 0. & (63) \end{cases}$$

Writting the solution

$$c(\rho) = T_1 e^{-\varepsilon(\mu+1)\rho} + T_2 e^{\varepsilon(\mu-1)\rho},\ (\ \mu = \sqrt{z^2 + \frac{7}{4}}\ \text{-dir}\ )$$

of the equation (62) in the first boundary conditions, we determine the characteristic equation

$$\Delta_4(z;\varepsilon) = \left(\frac{9}{4} - z^2\right) sh\left(\varepsilon\sqrt{4z^2 + 7}\right) = 0. \tag{64}$$

Theorem 4. The set of roots of the equation (64) consists of the denumerable number roots of order

$$z_n^{\pm} = \pm i\sqrt{\frac{7}{4} + \frac{\pi^2 n^2}{4\varepsilon^2}} \qquad (n = 1,2,...)$$

and $z = \pm\dfrac{3}{2}$.

The solutions corresponding to the determined roots of the characteristic equation (64) are determined as folowas:

a) $z = -\dfrac{3}{2}$

$$u_\varphi^{(1)}(\rho;\theta) = C_0 e^{\varepsilon\rho}\left(\frac{1}{2}\sin\theta \ln\left(ctg^2\left(\frac{\theta}{2}\right)\right) + ctg\,\theta\right), \tag{65}$$

в) $z_n^{\pm} = \pm i\sqrt{\frac{7}{4} + \frac{\pi^2 n^2}{4\varepsilon^2}}$

$$u_{\varphi}^{(2)}(\rho;\theta) = \sum_{n=1}^{\infty} e^{-\varepsilon(1+\rho)}\left[4\sin\left(\frac{\pi n}{2}(1-\rho)\right) - \frac{\pi n}{\varepsilon}\cos\left(\frac{\pi n}{2}(1-\rho)\right)\right] m_n'(\theta), \quad (66)$$

The general solution of the problem (3),(6) are determined by the formula

$$u_{\varphi}(\rho;\theta) = C_0 e^{\varepsilon\rho}\left(\frac{1}{2}\sin\theta \ln\left(ctg^2\left(\frac{\theta}{2}\right)\right) + ctg\,\theta\right) +$$
$$+\sum_{n=1}^{\infty} e^{-\varepsilon(1+\rho)}\left[4\sin\left(\frac{\pi n}{2}(1-\rho)\right) - \frac{\pi n}{\varepsilon}\cos\left(\frac{\pi n}{2}(1-\rho)\right)\right] m_n'(\theta), \quad (67)$$

For the torsional moment $M_{bur.}$ of stress acting on the section $\theta = const$ we obtain:

$$M_{bur} = -2\pi G_0 sh(4\varepsilon) C_0, \quad (68)$$

When the lateral surface of the sphere is free from stress, the constant C_0 is equivalent to the torsional moment $M_{bur.}$ of stresses acting on the section $\theta = const$.

The solution (65) determines the internal stress-strain state of the sphere. The stress state corresponding to the solution (66) is of boundary layer character and the first term of its asymptotic expansion is equivalent to the Saint-Venant baundary effect in theory of inhomogeneous plates.

When going a way from the conical sections $\theta = \theta_j$ $(j = 1;2)$ the solution (66) damps exponentially. According to (68)

$$C_0 = -\frac{M_{bur}}{2\pi G_0 sh(4\varepsilon)}$$

As a result of satisfaction of the boundary conditions (8) given on the bases, we obtain a system of linear equations:

$$m_k'(\theta_s) ctg\,\theta_s - \left(2 + \frac{\pi^2 k^2}{4\varepsilon^2}\right) m_k(\theta_s) = \frac{\varepsilon^2}{G_0(16\varepsilon^2 + \pi^2 k^2)} \int_{-1}^{1} f_s^*(\rho) e^{\varepsilon(\rho+1)} \times$$

$$\times \left[\frac{\pi k}{\varepsilon} \cos\left(\frac{\pi k}{2}(1-\rho) \right) - 4 \sin\left(\frac{\pi k}{2}(1-\rho) \right) \right] d\rho, \qquad (69)$$

here $m_k(\theta) = A_k^{(1)} P_{z_k - \frac{1}{2}}(\cos\theta) + A_k^{(2)} Q_{z_k - \frac{1}{2}}(\cos\theta)$; $P_{z_k - \frac{1}{2}}(\cos\theta)$ and $Q_{z_k - \frac{1}{2}}(\cos\theta)$ are Legendre function of first and second kind, respectively; $f_s^*(\rho) = f_s(\rho) - \dfrac{e^{\varepsilon\rho} M_{bur.}}{\pi \sin^2\theta_s sh(4\varepsilon)}$; $(s = 1;2)$.

The constants $A_k^{(1)}$, $A_k^{(2)}$ are determined from the system (69).

In **2.2** a torsional problem for a radial inhomogeneous small thickness sphere closed from the lateral surface is studied. A boundary layer character homogeneous solution is built. The character of the stress-strain state is determined

In **2.3** a torsional problem of a sphere whose elasticity modulus is a radius dependent arbitrary continuous function and whose lateral surface is free from stresses, is studied.

In **2.4** a torsional problem of a small thickness sphere with closed lateral surface whose elasticity modulus is a radius dependent arbitrary continuous function is studied.

In **2.5** a torsional vibration problem of a radial inhomogeneous sphere with no 0 and π poles, is considered. It is assumed that the G elasticity modulus and the density of the material of g sphere with respect to the radius changes by the linear law

$$G(r) = G_* r, g = g_* r \ .$$

The expression of the equation of motion expressing the torsional vibrations of the sphere in the spherical coordinate system in the displacements is as follows:

$$\frac{\partial^2 u_\varphi}{\partial \rho^2}+\frac{3}{\rho}\left(\frac{\partial u_\varphi}{\partial \rho}-\frac{u_\varphi}{\rho}\right)+$$
$$+\frac{1}{\rho^2}\left(\frac{\partial^2 u_\varphi}{\partial \theta^2}+\frac{\partial u_\varphi}{\partial \theta}ctg\,\theta-\frac{\cos 2\theta}{\sin^2\theta}u_\varphi\right)=\frac{g_* r_0^2}{G_*}\frac{\partial^2 u_\varphi}{\partial t^2}, \tag{70}$$

here $\rho=\frac{r}{r_0}$ -is a pure variable; $r_0=\frac{r_1+r_2}{2}$.

It is assumed that the lateral surface of the sphere is free from load.

The solution of the equation (70) is sought in the form of

$$u_\varphi(\rho,\theta,t)=v(\rho)m(\theta)e^{i\omega t} \tag{71}$$

Writing (71) in (70) and in the boundary condition given in the lateral surface, we obtain the boundary value problem

$$\begin{cases} v''(\rho)+\dfrac{3}{\rho}v'(\rho)+\left(\Omega^2-\dfrac{\left(z^2+\frac{3}{4}\right)}{\rho^2}\right)v(\rho)=0, & (72)\\[2ex] \left.(\rho v'(\rho)-v(\rho))\right|_{\rho=\rho_s}=0, & (73)\end{cases}$$

Here $\Omega^2=\frac{g_*\omega^2 r_0^2}{G_*}$ is a pure frequency; $s=1;2$.

Writing the solution $v(\rho)=\rho^{-1}\left[J_{\sqrt{z^2+\frac{7}{4}}}(\Omega\rho)A+Y_{\sqrt{z^2+\frac{7}{4}}}(\Omega\rho)B\right]$ of the equation (72) in the boundary conditions (73) from the existence of nontrivial solution of the system of linear algebraic equations we determine the dispersion equation

$$\Delta(z,\Omega,\rho_1,\rho_2)=4L^{(0;0)}_{\sqrt{z^2+\frac{7}{4}}}(\Omega\rho_1;\Omega\rho_2)-2\Omega\left[\rho_1 L^{(1;0)}_{\sqrt{z^2+\frac{7}{4}}}(\Omega\rho_1;\Omega\rho_2)+\right.$$

$$\left.+\rho_2 L^{(0;1)}_{\sqrt{z^2+\frac{7}{4}}}(\Omega\rho_1;\Omega\rho_2)\right]+\Omega^2\rho_1\rho_2 L^{(1;1)}_{\sqrt{z^2+\frac{7}{4}}}(\Omega\rho_1;\Omega\rho_2)=0, \quad (74)$$

here

$$L^{(s;j)}(\Omega\rho_1;\Omega\rho_2)=L^{(s)}_{\sqrt{z^2+\frac{7}{4}}}(\Omega\rho_1)Y^{(j)}_{\sqrt{z^2+\frac{7}{4}}}(\Omega\rho_2)-J^{(j)}_{\sqrt{z^2+\frac{7}{4}}}(\Omega\rho_2)\times$$

$\times Y^{(s)}_{\sqrt{z^2+\frac{7}{4}}}(\Omega\rho_1)$; $(s=0;1;\ j=0;1)-\text{dir};$ $J_{\sqrt{z^2+\frac{7}{4}}}(\Omega\rho)$, $Y_{\sqrt{z^2+\frac{7}{4}}}(\Omega\rho)$ -are first and second kind Bessel functions.

As $\varepsilon\to 0$ for Ω satisfying the condition $\Omega=O(1)$ the dispersion equation (74) has the limited roots

$$z_k=z_{0k}+\varepsilon z_{k2}+....(k=1;2).$$

As $\varepsilon\to 0$ for the frequencies satisfying the condition $\Omega\to\infty$ the roots of the dispersion function increase infinitely.

1) As $\Omega\to\infty$ and $\varepsilon\Omega\to 0(\varepsilon\to 0)$ the dispersion equation has the roots:

a) $$z_k=z_{k0}\varepsilon^{-\beta},z_{k0}=O(1),\quad 0<\beta<1$$

$$z_k=z_{k0}\varepsilon^{-\beta}+z_{k2}\varepsilon^{\beta}+...;\quad 0<\beta<\frac{1}{2}.$$

$$z_k=z_{k0}\varepsilon^{-\beta}+z_{k2}\varepsilon^{2-3\beta}+...;\quad \frac{1}{2}<\beta<1.$$

b) $$z_k=i\left(\frac{\delta_k}{\varepsilon}+O(\varepsilon^{1-\beta})\right);\ 0<\beta<1\ ,\sin(2\delta_k)=0.$$

2) As $\Omega\to\infty$ and $\varepsilon\Omega\to const$ ($\Omega=\frac{\Omega_0}{\varepsilon}$, $\Omega_0=O(1)$) the dispersion equation has the roots

$$z_k = \frac{\gamma_k}{\varepsilon} + O(\varepsilon)\ ,\ \sin\left(2\sqrt{\Omega_0^2 - \gamma_k^2}\right) = 0.$$

Asymptotic expressions for the homogeneous corresponding to the determined roots of the dispersion equation are deteremined.

In **2.6** torsional vibration of a radial inhomogeneous solutions small thickness sphere with closed lateral surface is studied. The roots of the dispersion equation

$$L^{(0;0)}_{\sqrt{z^2+\frac{7}{4}}}(\Omega\rho_1;\Omega\rho_2) = 0,$$

obtained from the fulfilment of homogeneous boundary conditions given in the lateral surface are asymptotically analyzed. Asymptotic expressions for displacement and stress tensor components corresponding to these roots, are built.

CONCLUSION

The dissertation work was devoted to the study of stress-strain state of a small thickness inhomogeneous sphere on the basis of elasticity theory equations. The following results were obtained:

1. The problem of symmetricity with respect to the axis of elasticity theory are studied when various boundary conditions are given on the lateral surface of a small thickness inhomogeneous sphere whose elasticity modulus change with respect to the radius by a linear law. As a result of asymptotic analysis, the character of the stress-strain state of the sphere was determined. Asymptotic formulas enabling to calculate the stress-strain state of the sphere to the required exactness are obtained. A new class of solutions that can not be described by any applied theory (boundary layer character solutions) are determined.
2. The torsion problem for a small inhomogeneous sphere was studied when various boundary conditions are given on the lateral surface. The methods enabling to build inhomogeneous solutions, are worked out.
3. Torsional vibration of a radial inhomogeneous small thickness sphere is studied. Simple asymptotic expressions for determining the displacement and stresses at various values of frequency are determined.

The basic results of the dissertation work are in the following works:

1. Hasanova, N.S. The torsional vibrations of a radially inhomogeneous sphere with closed lateral surface. // - Problems of application of mathematics and new information technologies, materials of the III Republican scientific conference, -Sumgayit: -2021, -p. 69-72.
2. Akhmedov, N.K., Gasanova, N.S., The problem of torsion of a spherical shell with variable shear moduli with a fixed side surface. // "Theoretical and applied problems of mathematics", materials of the international conference, Sumgayit - 2017, -p. 127-129.
3. Akhmedov, N.K., Gasanova, N.S. Asymptotic behavior of the solution of the problem of elasticity theory for a radially inhomogeneous sphere of small thickness. //-1st international science and engineering conference. Baku Engineering University, 2018 (29-30 november), -Baku: 2018. - p.94-96
4. Gasanova, N.S. Analysis of the problem of torsion for a radially inhomogeneous spherical belt with a fixed side surface. // -ВİСНИК Національного технічного унiверсiтету, «ХПİ», № 49(1221) -2016, -p.3-7.
5. Akhmedov, N.K., Gasanova, N.S. Studying the problem of torsion of a spherical shell with variable shear module, //-Transactions of NAS of Azerbaijan, Issue Mechanics, XXXVI, № 7 -2016, p. 3-10.
6. Akhmedov, N.K., Gasanova, N.S. Asymptotic behavior of the solution of an axisymmetric problem of elasticity theory for a sphere with variable elasticity modules //- Mathematics and Mechanic of Solids, 2019, 25 (12), -p. 2231-2251.
7. Gasanova, N.S. Analysis of elasticity theory problem for a radially inhomogeneous sphere with fixed lateral surface" //- Transactions of NAS of Azerbaijan, Issue Mechanics, -2019. 39(8), p.8-16.
8. Gasanova, N.S. Analysis of the axisymmetric problem of the theory of elasticity for the a radial inhomogeneous sphere"//- XXXIX

International Scientific – Practical conference “Advances in Science and Technology -2021. p. 45-50.
9. Gasanova, N.S. Analysis of axially – symmetric problem of elasticity theory for a radially inhomogeneous sphere under mixed boundary conditions on lateral surfaces, //- Transactions of NAS of Azerbaijan, ser. Phys. – Tech. – 2021. 41(7), p.8-14.

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